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Discrete-continuous computational model of the coupled dynamic system: pantograph – overhead contact line

Abstract: The paper presents the computational model of the pantograph – overhead contact line (OCL), which uses the theory of cable vibrations and Lagrange – Ritz approximation method to derive equations of motion of the overhead contact line subjected to moving pantographs. The pantograph is modelled as a dynamic system of two degrees of freedom describing the motion of two masses replacing a collector head and an articulating frame. The overhead contact line is defined as a catenary system with continuously distributed mass. It consists of a multi-span cable characterized by a curvilinear route (catenary wire) and a straight cable (contact wire) connected with a catenary wire by elastic droppers. The main objective of the paper is to present principal ideas of the computational model, with a particular emphasis on formulating the equation of motion of a pre-tensioned multi-span cable with non-negligible static sag. Much attention is paid to the description of dynamic interaction between the pantograph and overhead contact line. The model allows computer simulation of catenary vibrations induced by two pantographs passing over the contact line, as well as a simulation of dynamic increments of the contact force.

Keywords: Overhead contact line; Pantograph; Computational model; Dynamics; Cable structures; Inertial moving loads

Introduction

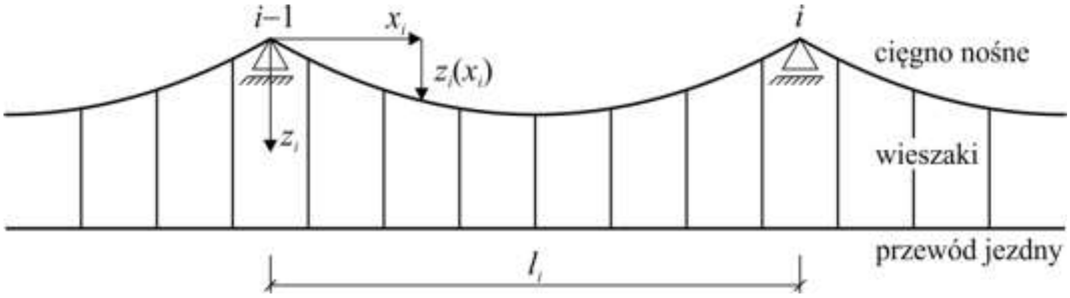
Theoretical studies of the dynamic interaction between pantograph and overhead contact line become important in connection with the requirements of technical specifications for interoperability (TSI) subsystem "Energy" of the rail system in the European Union, introduced by Commission Regulation (EU) from 2014 year [10]. The overhead contact line (OCL) is one of components of the interoperability of the rail system and as such is subjected to conformity assessment. According to the point 6.1.4 of the Annex to the mentioned Regulation, the construction of the contact line should be assessed inter alia on the basis of numerical simulation of lattice vibration caused by the use of pantographs mounted on railway vehicles, using a suitable calculation tool. Specific requirements for computational tools (simulation methods) are specified in PN-EN 50318: 2003 [9]. According to these requirements, the method of simulation must take into account the dynamic interaction between the pantograph and the supporting system of the OCL. The simulations should include courses of changes in contact forces on pantographs, waveforms of vibration of the OCL at selected points of the analysed section of the network and the history of the vibration of the main components of the pantograph during its passing along the trolley wire. Dynamic characteristics of the OCL - pantograph system, determined on the basis of the simulation are analysed to investigate whether they are within the scope of the more detailed regulations, ensuring proper quality of current collection.

Development of an effective and correct calculation tool for the numerical simulation is not an easy issue from the theoretical point of view. This is mainly due to the specific nature of the mechanical work of string structures, which include carrier system of overhead contact line [4]. The fundamental difficulty is, however, a theoretical solution of the problem of complex string structure vibration loaded by movement of pantographs, or subjected to movable inertial load with its own degrees of freedom. This theme is taken up in recent years in Polish literature, but computing models presented in the publications are generally highly simplified from the point of view of the accuracy of modelling OCL. For example, in [5], the authors treated the carrier system of traction network as a dynamic system with one dynamic degree of freedom with variable parameters approximated by harmonic functions of time. A similar approach is applied by authors of papers [7] and [6] operating the reduced mass and rigidity of OCL, as variable parameters in the area of the span. More precise model of the OCL , but far from the actual conditions, was used in [11], where the contact wire was modelled by finite element method using a beam elements, so with the regard of bending, whose influence is negligible in the case of a flaccid cable.

In all these publications, a lot of attention was paid on the problem of modelling the pantograph, which to some extent justifies a significant simplification of the traction network model. This paper proposes a very different, more "sustainable" computational model OCL - pantograph, where the emphasis is on the correct modelling network traction and dynamic interactions of a pantograph and overhead contact line. The pantograph is modelled as a discrete dynamic system with two degrees of freedom that determines the motion of the two masses to replace a collector head and articulating frame. The carrier catenary was treated as a flat system with continuous distribution of mass, consisting of multi-span flaccid cable with curvilinear route (catenary wire) and straight cable (contact wire) connected to it via the elastic droppers. The main aim of this work is to present the main ideas of the computational model, with particular emphasis on output equations of the dynamics of multi-span cable with non-negligible static sag. Numerical tests and application of the model will be the subject of a separate publication.

Basic assumptions of the computational model

The considered dynamic system consists of two main subsystems: overhead contact line (OCL) of chain type with a contact wire and one or two pantographs spaced a distance d and moving with a constant equal speed υ. The carrier system of OCL constitutes the pre-tensioned wire catenary and contact wire overhead to catenary wire by flaccid droppers of wire type. The model of support system includes one OCL, consisting of several sections. A diagram of a sample section (span) of the network is shown in Figure 1. The network model is flat, linear in the geometric and physical sense. It is designed for the analysis of vertical vibration of contact and catenary wire and the collector head of the pantograph, as well as changes of contact force in the course of time. It takes into account the strength of the interaction between the subsystems, the influence of sag on elastic responses of catenary wire and continuous distribution of weight of line and contact wire.

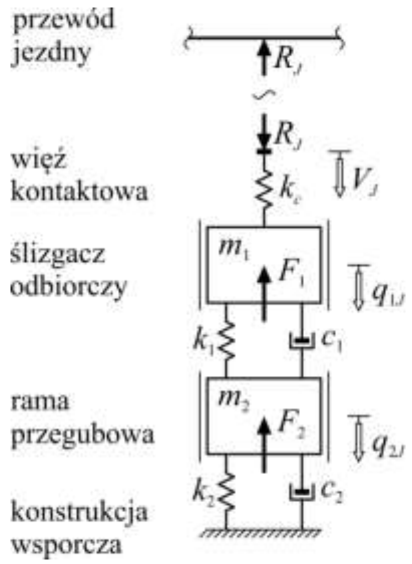


1. Diagram of the selected segment of overhead contact line

It was assumed that the catenary wire is a multi-span, based with possible sliding on vertically deformable supports, located on the same level. The number of spans of catenary wire is arbitrary and is n, rope trail in the unloaded condition is parabolic within each span, and the support points on the tight rope on extreme supports of the section are fixed points. The mechanical model of catenary wire is a linear-elastic catenary flabby wire with axial stiffness E_cA_c and unit mass $m_c = \rho_c A_c$, where ρ_c is the bulk density of the rope material, A_c area of cross-section, and E_c Young's modulus. The initial level of tension of multi-span cable is known and is H_0. The model of contact wire is a horizontal string with stiffness E_pA_p and initial mass m_p, pre-tense with the constant force S. The string is suspended by droppers to the catenary wire and non-displaceably supported in the endpoints of section.

Flaccid droppers joining the contact wire with the catenary wire are treated as massless deformable elements, which do not carry compression, while at tensile, they behave as linear-elastic ties. They are therefore elastic elements of "bond alloy" type [3], which are characterized by a different stiffness, which depends on the extent of the relative movement of the ends of ties. It was assumed that the stiffness of ties modelling droppers of OCL is equal to zero in compression and in tension it is k_{ij}, where by index $i = 1, 2, \ldots, n$ the number of OCL spans was indicated, and by index $j = 1, 2, \ldots, h_i$ the number of droppers in span i. In practice, the sections of overhead contact line are generally reproducible within each section - it means that the length of the spans l_i is the same and the number of droppers h_i in spans and stiffness of droppers k_{ij}. These parameters were taken as variables, in order to maintain generality of the model.

According to the guidelines of Appendix A of the European Standard EN 50318 [9], the pantograph model was adopted in the form of discrete dynamic system with two degrees of freedom, whose diagram as shown in Figure 2. The system consists of two masses representing the collector head and articulating frame of the pantograph, which are connected with each other and with the support structure (vehicle) by the system of elastic ties and hyper viscous dampers. In addition, a so-called contact spring was introduced between the collector head and the point of contact of the contact pads of the contact wire. The contact spring is not a part of the pantograph, but its use makes it easier to calculate the force of the contact pressure in the dynamic conditions, and is permitted by the standard EN 50318. The force of the contact pressure is indicated on Figure 2 by the symbol R_J and is time-variable response of the contact tie $R_J(t)$, index J = I, II is the number of pantograph. Reaction $R_J(t)$ includes both the static force of the contact pressure and the dynamic changes of the force caused by vibrations raised against passing the pantograph. The strength of the static pressure results from the operation of the pantograph lifting device and is directed upwards, thus transmits the weight of the pantograph. Depending on the design of the pantograph, the constant force rising the pantograph can be applied to the mass m_2 representing articulating frame or the mass of collector head m_1 or both masses. Forces F_1 and F_2 shown in Figure 2 are the respective resultant rising forces and gravity forces of the relevant parts of the pantograph.

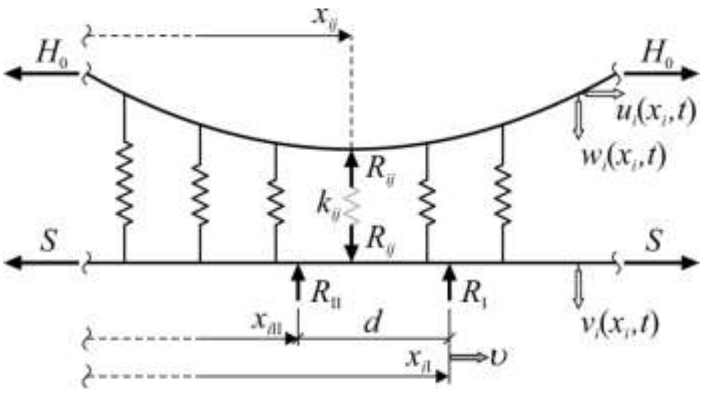


2. Dynamic model of pantograph

In the further part of this article, equations of motion will be derived in the matrix form, describing the following subsystems: catenary wire, contact wire and pantograph, with regard to the interaction forces between the subsystems. They will then be determined in explicit form of interaction strength, and on this basis will be formulated a target system of the coupled equations of motion describing the vibrations of the OCL section, excited by simultaneous passing two pantographs.

Catenary wire

Figure 3 shows the selected section of the OCL, marked by index *i*. Elastic ties (droppers) connecting the contact wire with catenary wire were replaced by their effects $R_{ij}(t)$, as shown in the example of the selected dropper with stiffness k_{ij}, whose location in the span *i* is determined by coordinate $x_i = x_{ij}$. In this way, wire and contact wire were separated, which allows the derivation of equation of rope motion, assuming its load by focused ties responses.



3. Displacement of catenary wire and contact wire and impact of droppers

As it is apparent from Figure 3, the vibrations of catenary wire are described by two functions: $u_i(x_i, t)$ and $w_i(x_i, t)$. The first function determines the horizontal component of the displacement, the second - the vertical component. Both movements are measured from the reference statically balanced state, wherein the catenary wire is pre-tensioned by horizontal force H_0 and burdened by its own distributed weight and the weight of contact wire and droppers transmitted by the droppers. The route of catenary wire in the baseline is described by the function $z_i(x_i)$ and its derivate $z_i' = \mathrm{d}z_i/\mathrm{d}x_i$ is determined by the angle of the slope of the tangent α_{0i}. Axial force in the cross-sectional x_i in the baseline is $N_{0i}(x_i) = H_0/\cos\alpha_{0i}$, which will be shown hereinafter, wherein $\cos\alpha_{0i} = (1 + z_i'^2)^{-1/2}$.

In the dynamic conditions, the route of catenary wire is deformed due to the load of passing pantograph and the catenary wire itself suffers from a longitudinal deformation, which with the assumption of small displacements are represented by the formula ([4], [1])

$$\varepsilon_i(x_i,t) = (u_i' + z_i'w_i')\cos^2\alpha_{0i} \tag{1}$$

According to Hooke's law, the time-varying longitudinal deformation of catenary wire cause changes in axial force in the catenary wire. The dynamic growth of the axial force caused by the deformation is

$$\Delta N_i(x_i,t) = E_\mathrm{c}A_\mathrm{c}\varepsilon_i = E_\mathrm{c}A_\mathrm{c}(u_i' + z_i'w_i')\cos^2\alpha_{0i} \tag{2}$$

The whole axial force $N_i(x_i, t) = N_{0i}(x_i) + \Delta N_i(x_i, t)$ meets the equation of dynamic equilibrium of catenary wire in the deformed configuration, which, according to [4], [1], [2] are expressed by

$$\begin{aligned} -[N_i\cos\alpha_{0i}(1+u_i')]' + (m_\mathrm{c}/\cos\alpha_{0i})\,\ddot{u}_i &= p_{xi} \\ -[N_i\cos\alpha_{0i}(z_i'+w_i')]' + (m_\mathrm{c}/\cos\alpha_{0i})\,\ddot{w}_i &= p_{zi} + q \end{aligned} \tag{3}$$

where $()' = \partial/\partial x$, $(\ddot{}) = \partial^2/\partial t^2$, $q(x)$ is distributed, constant load of catenary wire. Load vibration influencing the catenary wire are only vertical reactions of droppers, therefore, $p_{xi} = 0$ and

$$p_{zi}(x_i,t) = -\sum_{j=1}^{n_i} R_{ij}(t)\delta(x_i - x_{ij}) \tag{4}$$

where the symbol $\delta(\cdot)$ means Dirac's delta function . The forces that cause damping of catenary wire will be included after the discretization of the system.

In the state of reference, we have $p_{zi} = 0$, $u_i = 0$ and $w_i = 0$, therefore $\Delta N_i(x_i, t) = 0$. Taking into account these relationships in equations (3), we obtain equations of statically balanced reference configuration from which it follows that

$$
\begin{aligned}
-(N_{0i}\cos\alpha_{0i})' = 0 \quad &\rightarrow \quad N_{0i}\cos\alpha_{0i} = \text{const} = H_0 \\
-(N_{0i}\cos\alpha_{0i} z_i')' = q \quad &\rightarrow \quad z_i'' = -q / H_0
\end{aligned}
\tag{5}
$$

If we denote by m averaged and evenly distributed on the span length mass of catenary wire, droppers and contact wire, then $q = mg$ and the course of catenary wire will be parabolic, as it is in the model assumptions. Therefore, we obtain: $z_i = 4f_i(\xi_i - \xi_i^2)$, where the arrow of line stag is $f_i = mgl_i^2/8H_0$ and $\xi_i = x_i/l_i$.

After taking into account the equation (5), and assuming that $m_c/\cos\alpha_{0i} \approx m_c$, when stag of catenary wire is small, equations (3) can be written in the form:

$$
\begin{aligned}
-[H_0 u_i'' + (\Delta H_i)' + (\Delta H_i u_i')'] &= -m_c \ddot{u}_i \\
-[H_0 w_i'' + (\Delta H_i z_i')' + (\Delta H_i w_i')'] &= -m_c \ddot{w}_i + p_{zi}
\end{aligned}
\tag{6}
$$

where $\Delta H_i = \Delta N_i \cos\alpha_{0i}$, wherein the equation (2) should be replaced for ΔN_i. Equations (6) are conditions of the balance of forces acting on the catenary wire during vibrations (disregarding damping). The components on the left side of the equations are non-linear elastic reactions of catenary wire, whereas on the right are grouped inertial forces and the dynamic load. This fact was used to determine the formulas of the energy balance of a multi-span cable. The calculations omitted nonlinear components of the elastic response of catenary wire: $(\Delta H_i u_i')'$ i $(\Delta H_i w_i')'$ and boundary conditions and conditions of the balance of power at the intermediate supports were taken into account. Then the Lagrange-Ritz method [8] was applied to derive a matrix equations of motion specified in the function of time. Details of the derivation can be reproduced based on the procedure outlined in the article [2], which describes vibrations of multi-span cable of railway.

The general form of a matrix motion equation of multi-span cable of OCL pantograph is as follows:

$$
\mathbf{B}_{cc}\ddot{\mathbf{q}}_c(t) + \mathbf{C}_{cc}\dot{\mathbf{q}}_c(t) + \mathbf{K}_{cc}\mathbf{q}_c(t) = \mathbf{f}_c(t) \tag{7}
$$

The equation (7) has a block structure resulting from the construction of the vector of generalized coordinates: $\mathbf{q}_c = \text{col}(\mathbf{q}_{w1}, \mathbf{q}_{u1}, \ldots, \mathbf{q}_{wn}, \mathbf{q}_{un}, \mathbf{r}_c)$, which consists of a coordinates' vector of accepted approximation of the state of displacement of the cable, kinematically acceptable, expressed by the formulas

$$
\begin{aligned}
w_i(x_i, t) &= \mathbf{s}_i^{\mathrm{T}}(x_i)\mathbf{q}_{wi}(t) \\
u_i(x_i, t) &= \mathbf{s}_i^{\mathrm{T}}(x_i)\mathbf{q}_{ui}(t) + \mathbf{f}_i^{\mathrm{T}}(x_i)\mathbf{r}_c(t)
\end{aligned}
\tag{8}
$$

where $\mathbf{s}_i^{\mathrm{T}}(x_i) = [\sin \pi\xi_i, \sin 2\pi\xi_i, ..., \sin n_i\pi\xi_i]$ is vector of approximation functions, $\xi_i = x_i / l_i$, $i = 1,\ldots, n$. The second approximate vector is $\mathbf{f}_i^{\mathrm{T}}(x_i) = [(0)_1, ..., (1-\xi_i)_{i-1}, (\xi_i)_i, ..., (0)_{n-1}]$ if $i = 2,\ldots, n-1$. When $i = 1$ and $i = n$ it should be assumed, respectively: $\mathbf{f}_1^{\mathrm{T}} = [(\xi_1)_1, (0)_2, .., (0)_{n-1}]$, $\mathbf{f}_n^{\mathrm{T}} = [(0)_1, ..., (0)_{n-2}, (1-\xi_n)_{n-1}]$, where n is the number of spans of OCL sections. Coordinates grouped in the vector $\mathbf{r}_{\mathrm{c}}$ are unknown sliding of the cable on the following sliding supports (supports of OCL), numbered from 1 to $n-1$. In the equation (7) damping was taking into account according to the Rayleigh's model. Therefore, the damping matrix is a combination of the stiffness and matrix inertia: $\mathbf{C}_{\mathrm{cc}} = \kappa_{\mathrm{c}}\mathbf{K}_{\mathrm{cc}} + \mu_{\mathrm{c}}\mathbf{B}_{\mathrm{cc}}$.

Contact wire

Differential equations of vibration of contact wire in a span „i" result directly from the equations of catenary wire (6), if we take into account that the initial tension is S and the course of catenary wire is ideally rectilinear. Then, $z_i' = 0$, i.e. $\cos\alpha_{0i} = 1$ and $\Delta H_i = \Delta N_i = E_{\mathrm{p}}A_{\mathrm{p}}U_i'$, where $U_i(x_i, t)$ is a horizontal displacement of the string (contact wire). Denoting the vertical displacement by $v_i(x_i, t)$ we receive on the base (6), equation in the form:

$$\begin{aligned} &-[SU_i'' + E_{\mathrm{p}}A_{\mathrm{p}}U_i'' + E_{\mathrm{p}}A_{\mathrm{p}}(U_i'^2)'] = -m_{\mathrm{p}}\ddot{U}_i \\ &-[Sv_i'' + E_{\mathrm{p}}A_{\mathrm{p}}(U_i'v_i')'] = -m_{\mathrm{p}}\ddot{v}_i + p_{zi} \end{aligned} \tag{9}$$

Omission of nonlinear components leads to the separation of equations (9), which means that in the linear problem, axial vibrations may be omitted in the computational model of OCL because they are independent of the lateral vibrations. Considering the fact that the contact wire is charged by reactions of droppers $R_{ij}(t)$ and moving forces of the contact pressure of pantographs, $R_{\mathrm{I}}(t)$ and $R_{\mathrm{II}}(t)$, we obtained based on the equation $(9)_2$ and Fig. **3** the following equation of transverse vibrations of the cable

$$\begin{aligned} &-Sv_i'' + m_{\mathrm{p}}\ddot{v}_i = p_{zi} \\ &p_{zi} = \sum_{j=1}^{n_i} R_{ij}(t)\delta(x_i - x_{ij}) - \sum_{J=\mathrm{I}}^{\mathrm{II}} R_J(t)\delta(x_i - x_{iJ}) \end{aligned} \tag{10}$$

Preparation of energetic balance with consideration of boundary conditions and then application of the Lagrange - Ritz method leads to the equation of motion

$$\mathbf{B}_{\mathrm{pp}}\ddot{\mathbf{q}}_{\mathrm{p}}(t) + \mathbf{C}_{\mathrm{pp}}\dot{\mathbf{q}}_{\mathrm{p}}(t) + \mathbf{K}_{\mathrm{pp}}\mathbf{q}_{\mathrm{p}}(t) = \mathbf{f}_{\mathrm{p}}(t) \tag{11}$$

In which $\mathbf{C}_{\mathrm{pp}} = \kappa_{\mathrm{p}}\mathbf{K}_{\mathrm{pp}} + \mu_{\mathrm{p}}\mathbf{B}_{\mathrm{pp}}$, as in the case of catenary wire. The vector of generalized coordinates $\mathbf{q}_{\mathrm{p}} = \mathrm{col}(\mathbf{q}_{v1},\ldots, \mathbf{q}_{vn}, \mathbf{r}_{\mathrm{p}})$ has again block structure and consists of the coordinate vectors of approximation of transverse displacements, adopted in subsequent spans $i = 1, 2,\ldots, n$ according to the equation

$$v_i(x_i, t) = \mathbf{s}_i^{\mathrm{T}}(x_i)\mathbf{q}_{vi}(t) + \mathbf{f}_i^{\mathrm{T}}(x_i)\mathbf{r}_{\mathrm{p}}(t) \tag{12}$$

Due to the similarity of boundary conditions, the approximation of transverse displacements of the contact wire (12) has a structure similar to the approximation of horizontal displacements of catenary wire $(8)_2$.

Pantograph

Let us now consider a dynamic model of the pantograph, which is shown in Figure 2. The index J = I, II differentiates two pantographs moving along the contact wire. It was assumed that the technical details of both pantographs are the same and temporarily it was accepted that the analysed pantograph with the number J is placed i the span i. According to the actual state it was assumed that , the displacement of both masses of each pantograph, shown in Figure 2, $q_{1J}(t)$ i $q_{2J}(t)$, are measured in the state, when the pantograph is raised, i.e. all its deformable ties including the contact tie already suffered the initial displacement caused by constant forces F_1 and F_2. This means that the total force $F = F_1 + F_2$ which is the strength of the static pressure acts directly on the contact wire, so the reaction of the contact tie is $R_J = F + P_J(t)$, where $P_J(t)$ is a dynamic increase in the force of the contact pressure, caused by the relative movement of the ends of the contact tie. On the basis of the energy balance and Lagrange equations you can easily derive the equation of vibration of the pantograph on the basis of three coordinates: $q_{1J}(t)$, $q_{2J}(t)$ and $V_J(t)$, which are respectively: vertical displacements of masses m_1 and m_2 as well as vertical displacements of the contact point of the contact pads with the contact wire. These equations we write in the following block-matrix form:

$$\begin{bmatrix} \mathbf{B}_J & \mathbf{0} \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}}_J \\ \ddot{V}_J \end{bmatrix} + \begin{bmatrix} \mathbf{C}_J & \mathbf{0} \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}}_J \\ \dot{V}_J \end{bmatrix} + \begin{bmatrix} \mathbf{K}_J & -k_\mathrm{c}\mathbf{e} \\ -k_\mathrm{c}\mathbf{e}^\mathrm{T} & k_\mathrm{c} \end{bmatrix}\begin{bmatrix} \mathbf{q}_J \\ V_J \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ P_J \end{bmatrix} \tag{13}$$

where $\mathbf{q}_J = [q_{1J}, q_{2J}]^\mathrm{T}$, $\mathbf{e} = [1, 0]^\mathrm{T}$, $\mathbf{0} = [0, 0]^\mathrm{T}$ and

$$\mathbf{B}_J = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C}_J = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}, \quad \mathbf{K}_J = \begin{bmatrix} k_1 + k_\mathrm{c} & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \tag{14}$$

The block entry of three equations in the matrix form (13) allows their decomposition on two equations

$$\mathbf{B}_J\ddot{\mathbf{q}}_J(t) + \mathbf{C}_J\dot{\mathbf{q}}_J(t) + \mathbf{K}_J\mathbf{q}_J(t) = k_\mathrm{c}\mathbf{e}V_J(t) \tag{15}$$

$$-k_\mathrm{c}\mathbf{e}^\mathrm{T}\mathbf{q}_J(t) + k_\mathrm{c}V_J(t) = P_J(t) \tag{16}$$

from which the first equation describes the vibrations of the pantograph coupled to the vibrations of OCL by the component $k_\mathrm{c}\mathbf{e}V_J(t)$, and the second equation allows to calculate the strength of the dynamic interaction between pantograph and overhead contact line. The strength of the interaction is simultaneously dynamic growth of force of the contact pressure, which after developing a formula (16) can be written in a simpler form: $P_J(t) = k_\mathrm{c}V_J(t) - k_\mathrm{c}q_{1J}(t)$. If the pantograph is the time t in the span i, then displacement $V_J(t)$ of the contact point of the pantograph is equal to the dynamic deflection of string (contact wire) in the cross-section $x_i = x_{iJ}(t)$. Given the equation (12), we have

$$V_J(t) = v_i[x_{iJ}(t), t] = \mathbf{s}_i^{\mathrm{T}}[x_{iJ}(t)]\mathbf{q}_{vi}(t) + \mathbf{f}_i^{\mathrm{T}}[x_{iJ}(t)]\mathbf{r}_{\mathrm{p}}(t) \tag{17}$$

To take account of the language of mathematics that in the time t pantograph is located in the specific span i, the following approximation function definitions, tracing the location of the contact point, were introduced:

$$\tilde{\mathbf{s}}_{iJ}(t) = \begin{cases} \mathbf{s}_i^{\mathrm{T}}[x_{iJ}(t)], \text{ gdy } 0 < x_{iJ} \le l_i \\ \mathbf{0}, \text{ gdy } x_{iJ} \le 0 \text{ lub } x_{iJ} > l_i \end{cases}, \quad \tilde{\mathbf{f}}_{iJ}(t) = \begin{cases} \mathbf{f}_i^{\mathrm{T}}[x_{iJ}(t)], \text{ gdy } 0 < x_{iJ} \le l_i \\ \mathbf{0}, \text{ gdy } x_{iJ} \le 0 \text{ lub } x_{iJ} > l_i \end{cases} \tag{18}$$

Definitions (18) allow to save the so-called tracking movement of the following form

$$V_J(t) = \sum_{i=1}^{n} (\tilde{\mathbf{s}}_{iJ}^{\mathrm{T}} \mathbf{q}_{vi} + \tilde{\mathbf{f}}_{iJ}^{\mathrm{T}} \mathbf{r}_{\mathrm{p}}) \tag{19}$$

When pantographs move along the contact wire with the same constant speed, the location of the contact point of the first pantograph, measured in the relation to the starting point of the analysed sections OCL is defined by the function $x_{\mathrm{I}} = \upsilon t$. The location of contact point in the span i in the relation to the point x_i = is determined by the function $x_{i\mathrm{I}} = \upsilon t - L_{i-1}$, where $L_{i-1} = l_1 + l_2 + \ldots + l_{i-1}$, where in the first span (i = 1) it should be assumed that $L_0 = 0$. In the case of the second pantograph we have $x_{i\mathrm{II}} = x_{i\mathrm{I}} - d$.

Vibrations of the system composed of two pantographs (collector) with numbers J = I, II are described by two equations (15), which written together in the following matrix form

$$\mathbf{B}_{\mathrm{oo}}\ddot{\mathbf{q}}_{\mathbf{o}} + \mathbf{C}_{\mathrm{oo}}\dot{\mathbf{q}}_{\mathbf{o}} + \mathbf{K}_{\mathrm{oo}}\mathbf{q}_{\mathbf{o}} = \mathbf{f}_{\mathrm{o}} \tag{20}$$

where $\mathbf{q}_{\mathrm{o}} = \mathrm{col}(\mathbf{q}_{\mathrm{I}}, \mathbf{q}_{\mathrm{II}})$ and $\mathbf{f}_{\mathrm{o}} = \mathrm{col}(\mathbf{f}_{\mathrm{I}}, \mathbf{f}_{\mathrm{II}})$, whereas the matrices of the system $\mathbf{B}_{\mathrm{oo}}$, $\mathbf{C}_{\mathrm{oo}}$, $\mathbf{K}_{\mathrm{oo}}$ are block-diagonal, for example $\mathbf{B}_{\mathrm{oo}} = \mathrm{diag}(\mathbf{B}_{\mathrm{I}}, \mathbf{B}_{\mathrm{II}})$.

Interaction forces and load of subsystems

After replacement of the right side of the equations (15) by the equation (19) and the transformation using matrix arithmetic we obtain a formula defining the vector of dynamic loads of pantographs

$$\mathbf{f}_{\mathrm{o}}(t) = \tilde{\mathbf{K}}_{\mathrm{op}}(t)\mathbf{q}_{\mathrm{p}}(t) \tag{21}$$

The vector (21) expresses the influence of the vibrating contact wire on the pantographs by the contact spring, because it depends on the generalized coordinates $\mathbf{q}_{\mathrm{p}} = \mathrm{col}(\mathbf{q}_{v1}, \ldots, \mathbf{q}_{vn}, \mathbf{r}_{\mathrm{p}})$, resulting from the approximation of vibration of contact wire (12). The matrix defining the vector (21) is calculated from the formulas

$$\begin{aligned} &\tilde{\mathbf{K}}_{\mathrm{op}}(t) = \left[(\tilde{\mathbf{K}}_{\mathrm{op}})_1 \quad \ldots \quad (\tilde{\mathbf{K}}_{\mathrm{op}})_n \quad (\tilde{\mathbf{K}}_{\mathrm{op}})_r \right] \\ &(\tilde{\mathbf{K}}_{\mathrm{op}})_i = \mathbf{E}\{k_{\mathrm{c}}\}\tilde{\mathbf{S}}_i^{\mathrm{T}}, \quad (\tilde{\mathbf{K}}_{\mathrm{op}})_r = \sum_{i=1}^{n} \mathbf{E}\{k_{\mathrm{c}}\}\tilde{\mathbf{F}}_i^{\mathrm{T}} \end{aligned} \tag{22}$$

where

$$\tilde{\mathbf{S}}_i = \left[\tilde{\mathbf{s}}_{i\mathrm{I}}(t) \quad \tilde{\mathbf{s}}_{i\mathrm{II}}(t)\right], \quad \tilde{\mathbf{F}}_i = \left[\tilde{\mathbf{f}}_{i\mathrm{I}}(t) \quad \tilde{\mathbf{f}}_{i\mathrm{II}}(t)\right], \quad \mathbf{E} = \begin{bmatrix} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{e} \end{bmatrix} \tag{23}$$

and $\{k_\mathrm{c}\} = \mathrm{diag}(k_\mathrm{c}, k_\mathrm{c})$, as previously: $\mathbf{e} = [1, 0]^\mathrm{T}$, $\mathbf{0} = [0, 0]^\mathrm{T}$.

Having the formula (19) and the structure of the vector $\mathbf{q}_\mathrm{p}$ we can write in the explicit form, the combined forces of the contact pressure (16), static and dynamic, as

$$R_J(t) = F + P_J(t) = F + k_\mathrm{c}\left[\tilde{\mathbf{s}}_{1J}^\mathrm{T}, \quad ..., \quad \tilde{\mathbf{s}}_{nJ}^\mathrm{T}, \quad \sum\nolimits_{i=1}^{n} \tilde{\mathbf{f}}_{iJ}^\mathrm{T}\right]\mathbf{q}_\mathrm{p} - k_\mathrm{c}\mathbf{e}^\mathrm{T}\mathbf{q}_J \tag{24}$$

where $F = F_1 + F_2$. Forces $R_J(t)$, where J = I, II, exert a load on the contact wire, from which the first component of the vector of generalized loads results $\mathbf{f}_\mathrm{p} = \tilde{\mathbf{f}}_\mathrm{p} + \hat{\mathbf{f}}_\mathrm{p}$ in the equation of motion (11). The second component is a result of the interaction of droppers connecting the contact wire with catenary wire, wherein the reaction of j-th dropper with stiffness k_{ij}, located in span i, is determined by the relationship:

$$R_{ij} = k_{ij}[w_i(x_{ij}) - v_i(x_{ij})] = k_{ij}(\mathbf{s}_{ij}^\mathrm{T}\mathbf{q}_{wi} - \mathbf{s}_{ij}^\mathrm{T}\mathbf{q}_{vi} - \mathbf{f}_{ij}^\mathrm{T}\mathbf{r}_\mathrm{p}) \tag{25}$$

in which it was taken into account the approximation of the displacement state of OCL system, adopted by the formulas (8) and 1 (12), and the markings were introduced: $\mathbf{s}_{ij} = \mathbf{s}_i(x_{ij})$, $\mathbf{f}_{ij} = \mathbf{f}_i(x_{ij})$.

After substituting the forces of interaction (24) and (25) into the formula $(10)_2$ as well as calculating the load work on the movements of the contact wire

$$L = \sum_{i=1}^{n} \int_0^{l_i} v_i(x_i, t)\, p_{zi}(x_i, t)\, dx_i = \mathbf{q}_\mathrm{p}^\mathrm{T}\mathbf{f}_\mathrm{p} \tag{26}$$

we obtained a vector of generalized loads that can be written by the general formula

$$\begin{gathered} \mathbf{f}_\mathrm{p}(t) = \tilde{\mathbf{f}}_\mathrm{p}(t) + \hat{\mathbf{f}}_\mathrm{p}(t) = \\ \tilde{\mathbf{F}}_\mathrm{p}(t) - \tilde{\mathbf{K}}_\mathrm{pp}(t)\mathbf{q}_\mathrm{p}(t) + \tilde{\mathbf{K}}_\mathrm{po}(t)\mathbf{q}_\mathrm{o}(t) - \hat{\mathbf{K}}_\mathrm{pp}\mathbf{q}_\mathrm{p}(t) + \hat{\mathbf{K}}_\mathrm{pc}\mathbf{q}_\mathrm{c}(t) \end{gathered} \tag{27}$$

Where, for example: $\tilde{\mathbf{F}}_\mathrm{p}(t) = \mathrm{col}\,(\tilde{\mathbf{S}}_1\mathbf{F}, ..., \tilde{\mathbf{S}}_n\mathbf{F}, \sum\nolimits_{i=1}^{n} \tilde{\mathbf{F}}_i\mathbf{F})$, wherein $\mathbf{F} = [F_1 + F_2, F_1 + F_2]^\mathrm{T}$.

Similarly was determined the vector of generalized loads $\mathbf{f}_\mathrm{c}$ present in the equation of catenary wire motion (7), except that the catenary wire is loaded only by droppers reactions (Figure 3). The general form of the vector is defined by the formula

$$\mathbf{f}_\mathrm{c}(t) = -\hat{\mathbf{K}}_\mathrm{cc}\mathbf{q}_\mathrm{c}(t) + \hat{\mathbf{K}}_\mathrm{cp}\mathbf{q}_\mathrm{p}(t) \tag{28}$$

Equations of motion system: overhead contact line - pantographs

The motion equations of analysed three subsystems: catenary wire, contact wire and two pantographs together form a dynamic computational model of overhead contact line, along which the pantographs are moving at a constant speed. After taking into account the load vectors (21), (27) and (28), the equation of motion (7), (11) and (20) take the form

$$\begin{aligned}
&\mathbf{B}_{cc}\ddot{\mathbf{q}}_{c}(t)+\mathbf{C}_{cc}\dot{\mathbf{q}}_{c}(t)+\mathbf{K}_{cc}\mathbf{q}_{c}(t)=-\hat{\mathbf{K}}_{cc}\mathbf{q}_{c}(t)+\hat{\mathbf{K}}_{cp}\mathbf{q}_{p}(t)\\
&\mathbf{B}_{pp}\ddot{\mathbf{q}}_{p}(t)+\mathbf{C}_{pp}\dot{\mathbf{q}}_{p}(t)+\mathbf{K}_{pp}\mathbf{q}_{p}(t)=\\
&\qquad\tilde{\mathbf{F}}_{p}(t)-\tilde{\mathbf{K}}_{pp}(t)\mathbf{q}_{p}(t)+\tilde{\mathbf{K}}_{po}(t)\mathbf{q}_{o}(t)-\hat{\mathbf{K}}_{pp}\mathbf{q}_{p}(t)+\hat{\mathbf{K}}_{pc}\mathbf{q}_{c}(t)\\
&\mathbf{B}_{oo}\ddot{\mathbf{q}}_{\mathbf{o}}+\mathbf{C}_{oo}\dot{\mathbf{q}}_{\mathbf{o}}+\mathbf{K}_{oo}\mathbf{q}_{\mathbf{o}}=\tilde{\mathbf{K}}_{op}(t)\mathbf{q}_{p}(t)
\end{aligned} \tag{29}$$

which makes clear that the vibrations of the subsystems are coupled. The method of coupling is more noticeable after the conversion of equations (29) into a single matrix equation with block structure. The transforming consists in the introduction of the combined vector of generalized coordinates $\mathbf{q}=\mathrm{col}(\mathbf{q}_{c},\mathbf{q}_{p},\mathbf{q}_{o})$ and transfer to the left side of equations (29) all components that depend on the generalized coordinates as well as their velocity and acceleration. As a result of this, we obtain the equation

$$\begin{aligned}
&\begin{bmatrix}\mathbf{B}_{cc}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{B}_{pp}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{B}_{oo}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}_{c}\\ \ddot{\mathbf{q}}_{p}\\ \ddot{\mathbf{q}}_{o}\end{bmatrix}+\begin{bmatrix}\mathbf{C}_{cc}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{C}_{pp}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{C}_{oo}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}_{c}\\ \dot{\mathbf{q}}_{p}\\ \dot{\mathbf{q}}_{o}\end{bmatrix}+\\
&\qquad\begin{bmatrix}(\mathbf{K}_{cc}+\hat{\mathbf{K}}_{cc})&-\hat{\mathbf{K}}_{cp}&\mathbf{0}\\ -\hat{\mathbf{K}}_{pc}&(\mathbf{K}_{pp}+\hat{\mathbf{K}}_{pp}+\tilde{\mathbf{K}}_{pp})&-\tilde{\mathbf{K}}_{po}\\ \mathbf{0}&-\tilde{\mathbf{K}}_{op}&\mathbf{K}_{oo}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{c}\\ \mathbf{q}_{p}\\ \mathbf{q}_{o}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \tilde{\mathbf{F}}_{p}\\ \mathbf{0}\end{bmatrix}
\end{aligned} \tag{30}$$

which comes to the well-known system of linear equations of motion that recorded in the matrix notation have a form:

$$\mathbf{B}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}(t)\mathbf{q}(t)=\mathbf{f} \tag{31}$$

It is worth noting that this is a system of equations of motion with variable coefficients, wherein, by the application of the contact tie, the time-dependent coefficients are grouped in only some blocks of stiffness matrix, marked in (30) as tilde.

Summary

proposed computational model OCL with movable pantographs can be reduced to the equations (31). Solving such equations can be obtained by direct numerical integration, for example by recursive β-Newmark's method. However, the numerical difficulty is the pseudo-linearity of equations (31), visible in their basic structure (30). In fact, all the blocks of the stiffness matrix marked with an override cap are dependent on the state of displacement, or the generalized coordinates, because stiffness of droppers k_{ij} occurring in these blocks is zero under compression. Therefore, for the stiffness k_{ij} of each dropper should be substituted zero, if the relative displacement of their ends ($v_{ij}-w_{ij}$) is negative. In this way, the algorithm of integration of equations becomes recurrent-iterative algorithm, because the relative displacement of each dropper must be checked in each integration step and blocks of stiffness matrix should be corrected in the iterative procedure, depending on the stiffness of the droppers. Numerical simulation of vibration of overhead contact line and dynamic changes in the forces of the contact pressure of the pantographs in this situation is very time-consuming. Despite this disadvantage, it is noted that the computational model presented in the paper enables considerable shortening of calculation time compared to the standard commercial models created in MES systems. Then, it leads to a much smaller number of equations of motion, and as it is known, computational efforts associated with the direct numerical simulation of vibration are growing rapidly with an increase in the size of the task, which is measured just by the number of the equations of motion.

The proposed computational model, thanks to the continuous model of catenary wire and contact wire includes not only a continuous distribution of mass, but also a number of other specific features of catenary wire work. The model takes into account the impact of static sag of the curved catenary wire on its elastic reactions, with simultaneous and accurate mapping the static curve of stag. The model also includes the impact of horizontal components of the displacement of catenary wire and mass inertia of catenary wire associated with this movement, as well as horizontal shifts in catenary wire in support points. The model introduces in a natural way, i.e. according to the mechanics of catenary wire, pre-tension of catenary and contact wire. It is difficult to obtain a similar accuracy of modelling using FEM, because it requires the use of a number of finite elements of "catenary" type and it should be added that simplified "truss" rod elements are often used in practical calculations, which basically makes impossible to achieve a comparable accuracy of modelling.

Numerical tests of the presented computational model will be the subject of subsequent publications, including the verification of the model using the so-called reference model described in Appendix A of EN 50318 [9]. In the more distant future, it is planning validation of the developed method of simulation of OCL vibration, on the basis of vibration measurement of the catenary system of selected OCL section.

Source materials

[1] Bryja D., Deterministyczne i stochastyczne metody analizy drgań mostów wiszących, Oficyna Wydawnicza Politechniki Wrocławskiej, Wrocław 2005.
[2] Bryja D., Knawa M., Computational model of an inclined aerial ropeway and numerical method for analysing nonlinear cable-car interaction, Computers & Structures, Civil Comp Ltd. and Elsevier Ltd., vol. 89, p. 1895-1905, 2011.
[3] Frýba L., Dynamics of Railway Bridges, Academia Praha 1996.
[4] Hajduk J., Osiecki J., Ustroje cięgnowe, teoria i obliczanie, WNT, Warszawa, 1970.
[5] Jagiełło A. S., Dudzik M, Wpływ prędkości przemieszczenia się pantografu wzdłuż sieci jezdnej na układ odbierak prądu – sieć trakcyjna, Pomiary Automatyka Kontrola (PAK), vol. 59, nr 10, s. 1084– 1088, 2013.
[6] Judek S., Karwowski K., Mizan M. Wilk A., Modelowanie współpracy odbieraka prądu z siecią trakcyjną, Przegląd Elektrotechniczny, r. 91, nr 19, s. 248-253, 2015.
[7] Kaniewski M., Symulacja uniesienia przewodów jezdnych sieci trakcyjnej pod wpływem przejazdu wielu pantografów, Elektrotechnika - Czasopismo Techniczne, Wyd. Polit. Krakowskiej, r. 108, z. 13, s. 143-153, 2011
[8] Langer J., Dynamika Budowli, Wyd. Politechniki Wrocławskiej, Wrocław 1980.
[9] Norma PN-EN: 2003 „Zastosowania kolejowe - Systemy odbioru prądu - Walidacja symulacji oddziaływania dynamicznego pomiędzy pantografem a siecią jezdną górną”.

[10] Rozporządzenie Komisji (UE) Nr 1301/2014 z dnia 18 listopada 2014 roku w sprawie technicznych specyfikacji interoperacyjności podsystemu „Energia” systemu kolei w Unii Europejskiej.
[11] Wątroba P., Duda S., Gąsiorek D., Symulacje numeryczne zjawisk dynamicznych w układzie pantograf – sieć jezdna, Modelowanie Inżynierskie, nr 54, s. 94-100, 2015.